SW Hydraulics

Useful for understanding flow with abrupt changes: fronts, gravity currents, sills

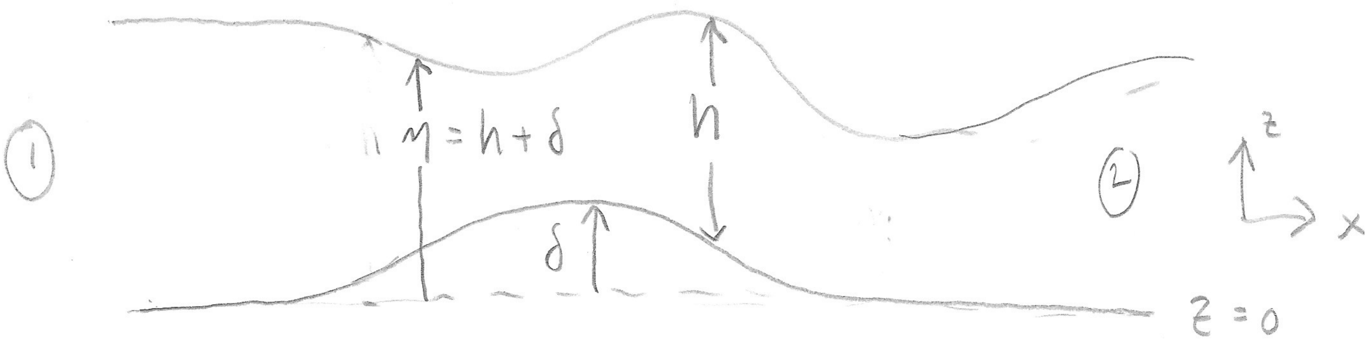
Consider:

- $\frac{a}{H}$ not $\ll 1$
- $\frac{U}{c}$ not $\ll 1$
- L set by topography, not wavelength

$\Rightarrow$ keep advection terms in SW equations

But, often OK to neglect $\frac{\partial}{\partial t}$ because local flow adjusts rapidly to a quasi-steady state.

RG Note: in the future include the $(\quad)h_x \approx \delta_x$ equation



transport / unit y

$\boxed{\text{mass}}$ $\cancel{\eta_t} + (h u)_x = 0 \Rightarrow hu = \text{const} = Q$

steady

$\boxed{x \text{ mom}}$ $\cancel{u_t} + u u_x + g \eta_x = 0$

$\Rightarrow \left(\frac{1}{2}u^2 + g\eta\right)_x = 0$ Bernoulli Function conserved (unless there is dissipation)

or $\left(\frac{1}{2}u^2 + gh + g\delta\right)_x = 0$

so integrating from an upstream location 1 where $\delta = 0$, $h = h_1$, $u = u_1$ to an arbitrary location over the bump:

$$\frac{1}{2}u^2 + gh + g\delta = \frac{1}{2}u_1^2 + gh_1 \quad (*)$$

noting $u = \frac{Q}{h} \Rightarrow \boxed{u^2 = \frac{Q^2}{h^2}}$ and the Froude number^2

is $\boxed{F^2 = \frac{u^2}{gh} = \frac{Q^2}{gh^3}}$

so we can rewrite (*) as

$$\frac{1}{2}\frac{Q^2}{h^2} + gh + g\delta = gh_1\left(\frac{1}{2}F_1^2 + 1\right)$$

we know Q, h_1, F_1

so this can be written as a cubic

to find $h(x)$ for a given $\delta(x)$

For relatively small bumps this gives:

"Subcritical"

$F_1 < 1$

and $F < 1$ for all x

"Supercritical"

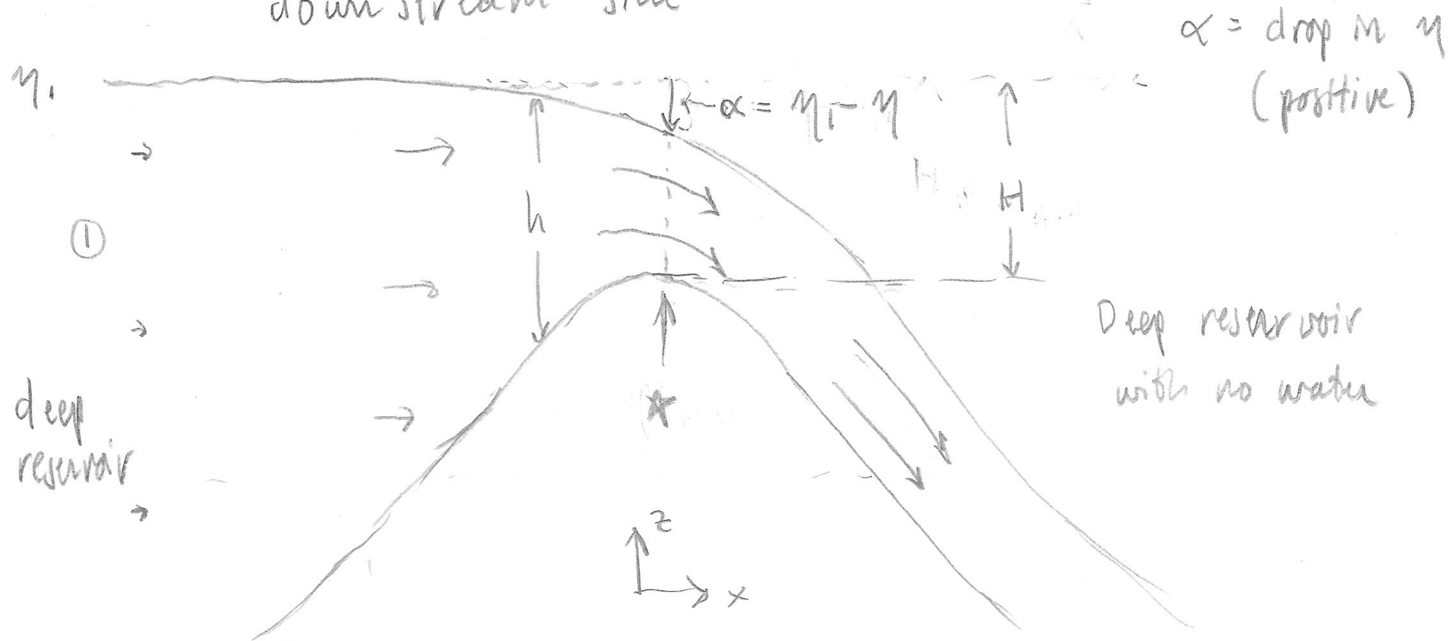
$F_1 > 1$

and $F > 1$ for all x

(*) What would you expect to see in $\langle \eta \rangle$ for a region like San Juan Channel

tidally averaged surface height

Consider flow over a bump where the water level is low on the downstream side



recall $Q = hu = \text{const.}$ $\boxed{\text{mass}}$

and $\left(\frac{1}{2}u^2 + g\eta\right)_x = 0$ $\boxed{x \text{ mom}}$ Bernoulli conserved

- upstream is very deep, so $u_1 \sim 0$ compared to u at ★

As surface height at ★ drops

u increases, but h decreases.

So what happens to transport hu?

★ At the peak

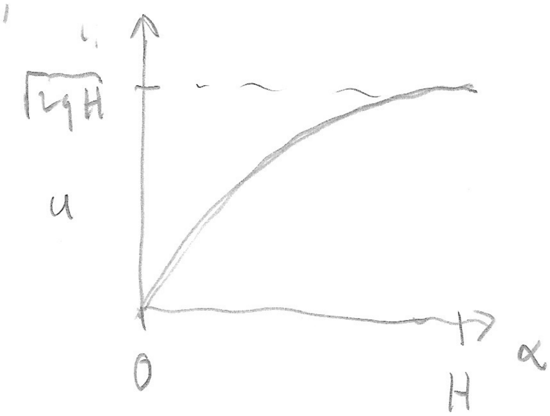
- find equations for u + h vs. α
- plot them for the possible range of α (0 to H)
- plot hu vs α
- at what value of α is hu a maximum?
- what is the Froude number at $hu = (hu)_{max}$?

My answers

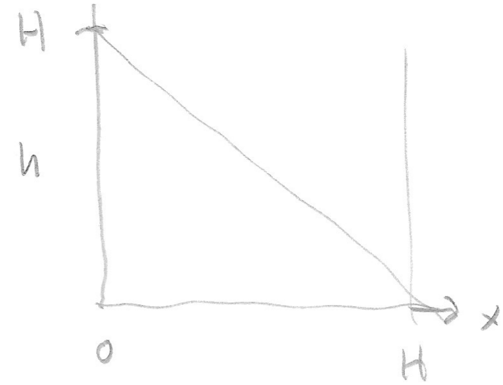
- $$\frac{1}{2}u^2 - g\eta = \overset{\ll \frac{1}{2}u^2}{\cancel{\frac{1}{2}u_1^2}} + g\eta_1$$

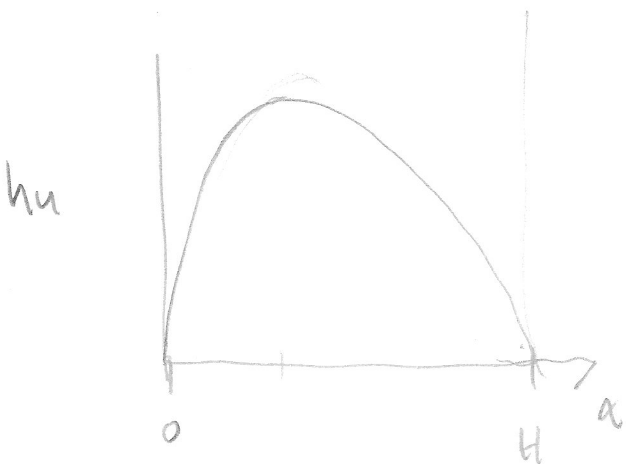
$$\frac{1}{2}u^2 = g(\eta_1 - \eta) = g\alpha$$

$$\Rightarrow \boxed{u = \sqrt{2g\alpha}}$$



- At peak $\boxed{h = H - \alpha}$





- to find α at $hu = (hu)_{max}$

take $\frac{\partial}{\partial\alpha}(hu) = 0 \Rightarrow \frac{\partial}{\partial\alpha}\left[(H-\alpha)\sqrt{2g\alpha}\right] = 0$

Answer continued:

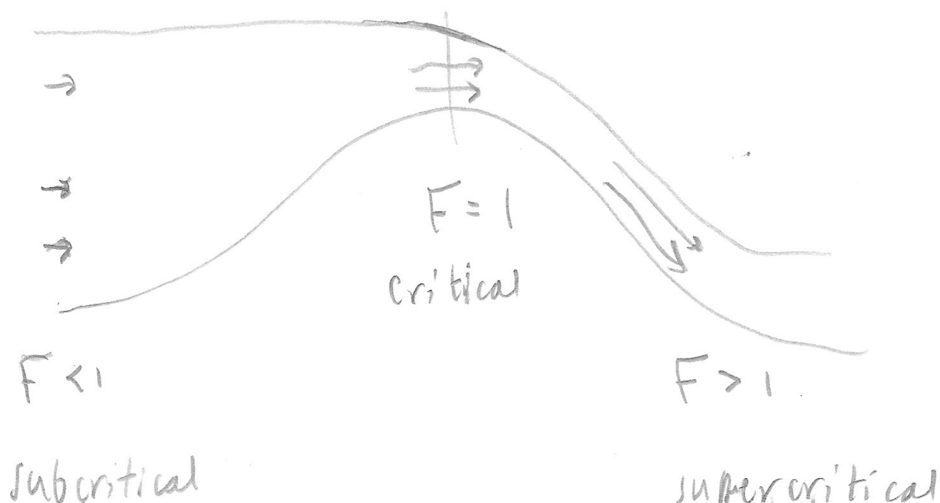
$$\sqrt{2g}\,\frac{d}{d\alpha}\left(H\alpha^{\frac{1}{2}} - \alpha^{3/2}\right) = 0$$

$$\Rightarrow \quad \frac{H}{2}\alpha^{-\frac{1}{2}} = \frac{3}{2}\alpha^{1/2} \quad \Rightarrow \boxed{\frac{H}{3} = \alpha} \Rightarrow \boxed{h = \frac{2}{3}H}$$

and $\quad u^2 = 2g\alpha = \frac{2}{3}gH$

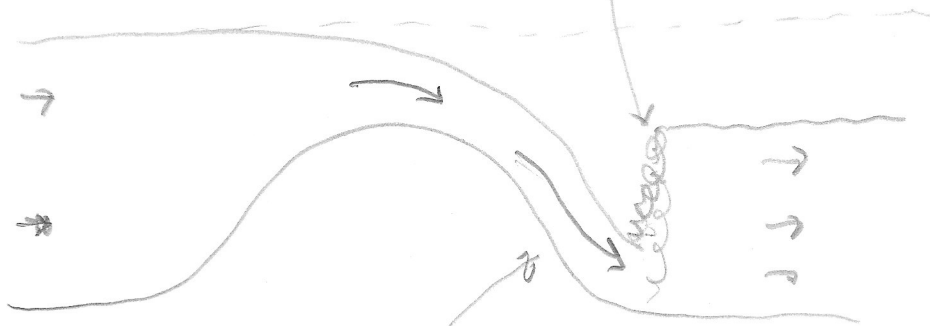
so $\quad F^2 = \frac{u^2}{gh} = \frac{\frac{2}{3}gH}{g\frac{2}{3}H} = 1 \quad \Rightarrow \boxed{F = 1}$

In this case F goes from sub- to supercritical and the solution looks like

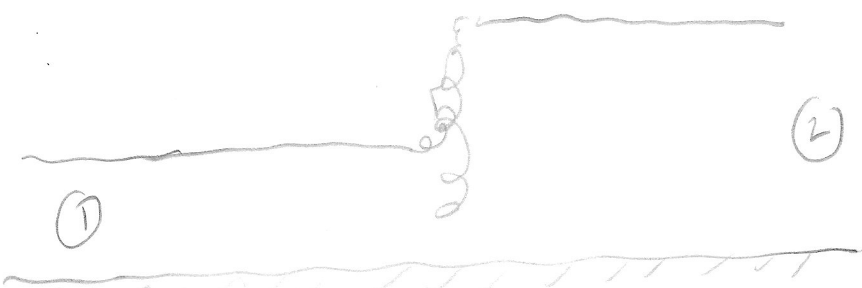


And it has the max possible transport

The other important solution is the "hydraulic jump."



$F > 1$ flow can abruptly transition to a turbulent "jump" with $F < 1$ on the other side



mass and momentum are conserved but not energy.

Can show $h_2 = \frac{h_1}{2}\sqrt{1 + 8F_1^2} - \frac{h_1}{2}$ ($h_2 > h_1$ for $F_1 > 1$)

and net $\frac{\text{energy loss}}{\text{unit width}} = \frac{\rho g Q}{4} \frac{(h_2 - h_1)^3}{h_1 h_2}$